SEQUENCES

A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

Sequences can be defined:

- by listing the first few elements
  e.g. 3, 5, 7, ...

- Analytically: by giving an explicit formula for its nth term
  e.g. $\forall n \in \mathbb{N}, a_n = \frac{(-1)^n}{n+1}$

- Recursively: using recursion
  e.g. $b_0 = 1, b_1 = 2$ (initial conditions)
  $b_k = b_{k-1} + b_{k-2}$ (recurrence relation)

RECURSIVELY DEFINED SEQUENCES

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is a fixed integer and k is any integer greater than or equal to i.

The initial conditions for such a recurrence relation specify the values $a_0, \ldots, a_{i-1}$.

SOLVING RECURRENCE RELATIONS BY ITERATION

- Look at $a_0, a_1, a_2, \ldots$ until you see a pattern. It is useful to not calculate the final value for each, but to keep the formula for each instead.
- Guess a formula for a_n
- Prove by induction that the formula is equivalent to the recursive definition

SUMMATION AND PRODUCT NOTATIONS

If m and n are integers and m≤n

The symbol $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$
The symbol (LHS) reads as "the sum from k equals m to n of a sub k"

The symbol $\prod_{k=m}^{n} a_k = a_m \times a_{m+1} \times \ldots \times a_n$
The symbol (LHS) reads as "the product from k equals m to n of a sub k"

The RHS is called the expanded form of the sum (or product).
k is called the index of the sum (or product)
m is the lower bound and n is the upper bound.

BASIC SEQUENCES AND THEIR SOLUTIONS

In all the following sequences you can assume that ∃a such that $a_0 = a$

Arithmetic sequences

A sequence $a_0, a_1, \ldots$ is called an arithmetic sequence iff

there is a constant c s.t. $a_k = a_{k-1} + c$ for all integers $k \geq 1$
This is equivalent to $a_n = a + c \,.\, n$ for all integers $n \geq 0$

Geometric Sequences

A sequence $a_0, a_1, \ldots$ is called a geometric sequence iff

there is a constant b s.t. $a_k = b \,.\, a_{k-1}$ for all integers $k \geq 1$
This is equivalent to $a_n = a \,.\, b^n$ for all integers $n \geq 0$

Sum of a Geometric Sequence

A sequence $a_0, a_1, \ldots$ is called a sum of a geometric sequence iff

there are constants b, c s.t. $a_k = b \,.\, a_{k-1} + c$ for all integers $k \geq 1$

This is equivalent to $a_n = b^n a + c\sum_{i=0}^{n-1} b^i$ for all integers $n \geq 0$

If a = c then $a_n = a\sum_{i=0}^{n} b^i$